Gelfand-Tsetlin theory

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Israel Moiseevich Gelfand (1913-2009)
Michael L’vovich Tsetlin (1924-1966)
Example
Assume the set \( \{1, a, b, c, \ldots\} \subset \mathbb{C} \) is linearly independent over \( \mathbb{Z} \)

\[
\begin{pmatrix}
1 & a+1 & a & b & 0 \\
a & b-1 & b & a+1 \\
c & c+1 & c \\
a & a-1 \\
a+1
\end{pmatrix}
\]

An element \( v \in \mathbb{C}^{\frac{n(n+1)}{2}} \)

\[
\begin{pmatrix}
a+1 & a & 1 & 0 & b \\
a & a & b-1 & b-1 \\
c & c & c \\
a-1 & a-1 \\
a & a-1 \\
a-1
\end{pmatrix}
\]

An element in normal form in \( S_\mu v \)

A seed in \( (\mathbb{Z}_0^\mu \# S_\mu) v \)
Fix a seed $\bar{v}$

$S_{\bar{v}}$ is the stabilizer of $\bar{v}$ in $G = S_1 \times \ldots \times S_n$

Above: $S_{\bar{v}} = S_1 \times S_2 \times S_3 \times (S_2 \times S_2)$

$$D(\bar{v}) = \{z \in \mathbb{Z}^{\frac{n(n-1)}{2}} | \bar{v} + z \text{ in normal form}\}$$

If $z \in D(\bar{v})$ then $(S_{\bar{v}})_z$ is the stabilizer of $z$ in $S_{\bar{v}}$

$S_{\bar{v}}^z$ the set of minimal length representatives of $S_{\bar{v}}/(S_{\bar{v}})_z$
Denote $I(\overline{v})$ the set of intervals corresponding to the blocks of the seed $\overline{v}$

In the example above: $I(\overline{v}) = \{(1)_1, (12)_2, (123)_3, (12)_4, (34)_4\}$

For $z \in D(\overline{v})$ denote $I(\overline{v}, z)$ the refinement of $I(\overline{v})$ corresponding to the blocks of $\overline{v} + z$

Then

$$\{D_\sigma(\overline{v} + z) \mid z \in D(\overline{v}), \sigma \in S^z_{\overline{v}}\}$$

is a $\mathbb{C}$-basis (of derivative tableaux) of the universal module $V(T(\overline{v})) = \mathcal{U}(\mathfrak{gl}_n)\chi$
For $\sigma \in S^z_v$ denote $D_{<\sigma}(\nu + z)$ an arbitrary linear combination of tableaux $D_{\tau}(\nu + z)$ with $\tau \in S^z_v$ strictly smaller than $\sigma$ in the induced Bruhat order.

$$l = [a, b]_k \subset l(\nu) \text{ with } k < n \text{ set}$$

$$e_l = \frac{\prod_{j=1}^{k+1} (x_{k,a} - x_{k+1,j})}{\prod_{(k,j) \notin l} (x_{k,a} - x_{k,j})}; \quad f_l = \frac{\prod_{j=1}^{k-1} (x_{k,b} - x_{k-1,j})}{\prod_{(k,j) \notin l} (x_{k,b} - x_{k,j})}$$
Theorem

The action of the canonical generators of $\mathfrak{gl}(n, \mathbb{C})$ on $V(T(\bar{v}))$ is given by the formulas

$$E_{k,k+1}D_\sigma(\bar{v} + z) =$$

$$= - \sum_{l \in \Pi(\bar{v}, z)[k]} \sum_{\tau \leq \sigma \alpha(l)} \mathcal{D}_{\tau, \sigma \alpha(l)}^{ar{v} + z}(e_l) D_\tau(\bar{v} + z + \delta^{k,a(l)}),$$

$$E_{k+1,k}D_\sigma(\bar{v} + z) =$$

$$= \sum_{l \in \Pi(\bar{v}, z)[k]} \sum_{\tau \leq \sigma \beta(l)} \mathcal{D}_{\tau, \sigma \beta(l)}^{ar{v} + z}(f_l) D_\tau(\bar{v} + z - \delta^{k,b(l)}),$$

$$E_{kk}D_\sigma(\bar{v} + z) = h_k(\bar{v} + z)D_\sigma(\bar{v} + z),$$

where $h_k = x_{k,1} + \cdots + x_{k,k} - (x_{k-1,1} + \cdots + x_{k-1,k-1}) + k - 1$, $\alpha(l) = (b, b - 1, \ldots, a)$, $\beta(l) = (a, a + 1, \ldots, b)$ and $\tau, \sigma$ are the Postnikov-Stanley operators.
Fix a complex vector space $V$, $\Lambda = S(V^*)$, $L$ the fraction field of $\Lambda$

$\Phi$ a finite root system with base $\Sigma$, $W = W(\Phi)$ the corresponding reflection group with minimal generating set $S$. Then $W$ acts on $\Lambda$ and $L$, set $\Gamma = \Lambda^W$

For $s \in W$,

$$\nabla_s = \frac{1}{\alpha_s} (1 - s) \in L^\#W,$$

a twisted derivation of $L$: for $f, g \in L$,

$$\nabla_s(fg) = \nabla_s(f)g + s(f)\nabla_s(g)$$
Example
Let $V = \mathbb{C}^2$ and $\{x, y\} \subset (\mathbb{C}^2)^*$ the dual basis to the canonical basis. Let $s$ the reflection given by $s(z_1, z_2) = (z_2, z_1)$, so $\alpha_s = x - y$. Then

$$\nabla_s(f)(x, y) = \frac{f(x, y) - f(y, x)}{x - y}$$

For $\sigma \in \mathcal{W}$ take a reduced decomposition $\sigma = s_1 \cdots s_\ell$ and set

$$\partial_\sigma = \nabla_{s_1} \circ \cdots \circ \nabla_{s_\ell}$$

$\nabla_s(\Lambda) \subset \Lambda$ for any $s \in S$
Set $\Delta(\Phi) = \prod_{\alpha \in \Phi^+} \alpha$, for each $\sigma \in \mathcal{W}$ set

$$S_\sigma = \frac{1}{|\mathcal{W}|} \partial_{\sigma^{-1}} \omega_0 \Delta(\Phi)$$

$\{S_\sigma \mid \sigma \in \mathcal{W}\}$ are Schubert polynomials, they form a basis of $\Lambda$ as a $\Gamma$-module, $\deg S_\sigma = \ell(\sigma)$

$$S_\sigma S_\tau = \sum_{\rho \in \mathcal{W}} c_{\sigma,\tau}^\rho S_\rho \mod I_\mathcal{W},$$

where $c_{\sigma,\tau}^\rho$ are the generalized Littlewood-Richardson coefficients, $I_\mathcal{W}$ the ideal of $\Lambda$ generated by the elements of $\Gamma$ of positive degree.
Consider $\Theta : \Lambda \to \text{Der}_\mathbb{C}(\Lambda)$, $\Theta(x_i) = \frac{\partial}{\partial x_i}$

Let $(-, -)_\Theta : \Lambda \times \Lambda \to \mathbb{C}$ the bilinear form given by $(f, g) = \Theta(f)(g)(0)$

$P_\sigma$ the unique element such that $(P_\sigma, S_\tau) = \delta_{\sigma, \tau}$ for all $\sigma, \tau \in W$, and $(P_\sigma, I_W)_\Theta = 0$

$\{P_\sigma \mid \sigma \in W\}$ Postnikov-Stanley polynomials

For $\sigma, \tau \in W$ with $\tau \leq \sigma$ in the Bruhat order set $D_\sigma = \Theta(P_\sigma)$.

Then

$$D_{\tau, \sigma} = \sum_{\rho \in \mathcal{W}} c^\sigma_{\tau, \rho} D_\rho$$
\[ \text{GT} \quad \text{the category of all Gelfand-Tsetlin } \mathfrak{gl}_n\text{-modules} \]

Fix \( \zeta \in \mathbb{C} \frac{n(n+1)}{2} / (\mathbb{Z} \frac{n(n-1)}{2} \# G) \), \( \bar{v} \in \zeta \)

\( \text{GT}_\zeta \) the full subcategory of \( \text{GT} \) of modules with support in \( \zeta \), i.e.
\( V \in \text{GT}_\zeta \) if
\[
V = \bigoplus_{z \in \mathbb{Z} \frac{n(n-1)}{2}} M[\bar{v} + z]
\]

\[
\text{GT} = \bigoplus_{\zeta \in \mathbb{C} \frac{n(n+1)}{2} / (\mathbb{Z} \frac{n(n-1)}{2} \# G)} \text{GT}_\zeta
\]
The F.-Ovsienko inequality:

$$\dim M[\bar{v} + z] \leq \frac{|S_v|}{|(S_{v})_z|}$$

Strong FO Conjecture:

*The set of all z such that equality above holds (the colored essential support) is nonempty*

hence the FO inequality gives a sharp bound in each subcategory GT$_\zeta$
Theorem (V.F., Grantcharov, Ramirez, Zadunaisky, 2018)

Let $\overline{v}$ be a seed in $\mathbb{C} \frac{n(n+1)}{2}$ and $\zeta = \zeta_{\overline{v}}$. Then

(i) The module $V(T(\overline{v}))$ has a simple socle $V_{\text{soc}}$

(ii) The Strong F.-Ovsienko Conjecture holds for $V_{\text{soc}}$

(iii) For any $\overline{v} + z$ in the essential support of $V_{\text{soc}}$, the module $V_{\text{soc}}$ is the unique simple Gelfand-Tsetlin module having $\overline{v} + z$ in its support

(iv) For any parabolic subgroup $G \subset S_{\overline{v}}$, $\frac{|S_{\overline{v}}|}{|G|}$ appears as a Gelfand-Tsetlin multiplicity in module $V_{\text{soc}}$
Recent developments

Following Webster:
Consider a smash product $\mathcal{F} = L \# W$ and $\mathcal{K} = L^W$

The standard flag order is the subalgebra

$$\mathcal{F}_\Lambda = \{ x \in \mathcal{F} | x(\Lambda) \subseteq \Lambda \}$$

A subalgebra $\mathcal{F} \subseteq \mathcal{F}_\Lambda$ is a principal flag order if $KF = \mathcal{F}_\Lambda$ and $W \subseteq F$

Then $F$ is a Galois order over $\Lambda$ ($G = \{1\}$, $M = M \rtimes W$)

Let $e = \frac{1}{|W|} \sum_{w \in W} w \in \mathcal{F}_\Lambda$. Then $\mathcal{K} \simeq e\mathcal{F}e$ and $\mathcal{K}_\Gamma \simeq e\mathcal{F}_\Lambda e$
For any flag order $F$, the centralizer algebra $U = eFe$ is a principal Galois order, and the category of $U$-modules is a quotient of the category of $F$-module.

Any principal Galois order $U$ has form $e\mathcal{F}_D e$ for some $\Lambda\#W \subset D \subset \mathcal{L}\#W$, where

$$\mathcal{F}_D = De \otimes_{\Gamma} U \otimes_{\Gamma} eD$$
Coulomb branches recently defined by Braverman, Finkelberg, Nakajima

A Coulomb branch is attached to each connected reductive complex group $G$ and representation $N$

Let $G[t]$ be the Taylor series points of the group $G$, $G((t))$ its Laurent series points and

$$Y = (G((t)) \times N[t])/G[t]$$

with a natural map $\pi : Y \to N[t]$

Let $H = N_{GL(N)}(G)$ the connected component of the identity in the normalizer of $G$, $T_H$ maximal torus, $Q \subset H$ the subgroup generated by $G$ and $T_H$
The Coulomb branch is the convolution algebra

\[ \mathcal{A} = H^Q_{\ast} (\pi^{-1}(N[t])) \]

There is a module structure on the \( Q \times \mathbb{C}^\ast \)-equivariant homology of any \( G[t] \)-invariant subvariety in \( N[t] \)

Let \( V = t^\ast_H \oplus \) where \( t_H \) is the Cartan Lie algebra of \( H \) and \( \mathcal{M} \) the cocharacter lattice of \( T_G \), acting by the \( h \)-scaled translations, \( W \) the Weyl subgroup of \( G \), \( \mathcal{M} \ltimes W \) the extended affine Weyl group.
Set

\[ \Lambda = T_{H^*} \times C^*(*) = S(t_H)[h] \]

\[ \Gamma = H_{Q^*} \times C^*(*) = S(t_H)^W[h] \]

\[ \Diamond \mathcal{A} \subset \mathcal{K}_\Gamma \text{ is a principal Galois order} \]
Gelfand-Tsetlin theory has strong connection with categorification theory and Khovanov-Lauda-Rouquier-Webster algebras

(Kamnitzer, Tingley, Webster, Weekes, Yacobi, 2018)
Invariants in symmetric algebra

\( \mathfrak{g} \) a simple Lie algebra over \( \mathbb{C} \) with basis \( Y_1, \ldots, Y_l \) s.t.

\[
[Y_i, Y_j] = \sum_k c_{ij}^k Y_k
\]

The adjoint action of \( \mathfrak{g} \) on itself extends to the symmetric algebra \( S(\mathfrak{g}) \) by

\[
Y(X_1 \ldots X_k) = \sum_{i=1}^k X_1 \ldots [Y, X_i] \ldots X_k
\]

The subalgebra of invariants is

\[
S(\mathfrak{g})^\mathfrak{g} = \{ P \in S(\mathfrak{g}) | Y(P) = 0 \text{ for all } Y \in \mathfrak{g} \}.
\]
Let $n = \text{rank} \mathfrak{g}$.

**Chevalley:** $S(\mathfrak{g})^g = \mathbb{C}[P_1, \ldots, P_n]$ for certain algebraically independent invariants $P_1, \ldots, P_n$ of certain degrees $d_1, \ldots, d_n$ depending on $\mathfrak{g}$.

For $\mathfrak{g} = \mathfrak{gl}_N$ set $E = (E_{ij})$
and write

$$det(u + E) = u^N + C_1 u^{N-1} + \ldots + C_N$$

Then $S(\mathfrak{gl}_N)^{\mathfrak{gl}_N} = \mathbb{C}[C_1, \ldots, C_N]$
Also $T_k = trE^k \in S(gl_N)^{gl_N}$ for all $k > 0$ and

$$S(gl_N)^{gl_N} = \mathbb{C}[T_1, \ldots, T_N]$$

$S(g)$ has the Lie–Poisson bracket $\{Y_i, Y_j\} = \sum_{k=1}^{l} c_{ij}^k Y_k$

$S(g)$ is Poisson commutative (Poisson centre of $S(g)$)
Problem: Extend $S(g)^g$ to a maximal Poisson commutative subalgebra of $S(g)$

Let $\mu \in g^*$. **Mischenko-Fomenko** subalgebra $\overline{A}_\mu$ is generated by the $\mu$-shifts of elements in $S(g)^g$, that is, by all the derivatives $D^j_\mu(p)$ for $p \in S(g)^g$ and $j \in \{0, \ldots, \deg p - 1\}$, where

$$D^j_\mu(p)(x) = \left. \frac{d^j}{dt} p(x + t\mu) \right|_{t=0}, \quad x \in g^*.$$

**Mischenko-Fomenko, 1978**: $\overline{A}_\mu$ is a Poisson-commutative subalgebra of $S(g)$.
Identify $\mathfrak{g} \simeq \mathfrak{g}^*$ via the Killing form. Let $Y_1, \ldots, Y_l$ be a basis of $\mathfrak{g}$.

Take $P = P(Y_1, \ldots, Y_l) \in S(\mathfrak{g})$ of degree $d$, $\mu \in \mathfrak{g}^*$. Substitute $Y_i \mapsto Y_i + z\mu(Y_i)$ and expand:

$$P(Y_1 + z\mu(Y_1), \ldots, Y_l + z\mu(Y_l)) = P(0) + P(1)z + \ldots + P(d)z^d$$

Then $\mathcal{A}_\mu$ is generated by all elements $P^{(i)}$ associated with all $\mathfrak{g}$-invariants $P \in S(\mathfrak{g})^\mathfrak{g}$

**Example**

$$\det(u + \mu + Ez^{-1}) \sum_{0 \leq i \leq k \leq N} C_k^i z^{-k+i} u^{N-k}$$

The elements $C_k^i$ with $k = 1, \ldots, N$ and $i = 0, 1, \ldots, k - 1$ are algebraically independent generators of $\mathcal{A}_\mu$ for regular $\mu$. 
\( \mu \in \mathfrak{g}^* \cong \mathfrak{g} \) is regular, if the centralizer \( \mathfrak{g}^\mu \) of \( \mu \) in \( \mathfrak{g} \) has minimal possible dimension = the rank of \( \mathfrak{g} \).

**Theorem**

If \( \mu \in \mathfrak{g}^* \) is regular, then

i) \( \overline{A}_\mu \) is maximal Poisson commutative

(A. Tarasov, 2002, for regular semisimple \( \mu \); Panyushev, Yakimova, 2008);

ii) \( P_k^{(i)} \) with \( k = 1, \ldots, n \) and \( i = 0, 1, \ldots, d_k - 1 \), are algebraically independent generators of \( \overline{A}_\mu \)

(colorblue Mischenko-Fomenko for regular semisimple \( \mu \);

Feigin, E.Frenkel and Toledano Laredo, 2010)
Vinberg’s problem

The universal enveloping algebra \( U(\mathfrak{g}) \) has a canonical filtration and

\[
gr \ U(\mathfrak{g}) \simeq S(\mathfrak{g}).
\]

Vinberg, 1990: find a commutative subalgebra \( \mathcal{A}_\mu \subseteq U(\mathfrak{g}) \) which “quantizes” \( \overline{\mathcal{A}}_\mu \), i.e. \( \text{gr} \ \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu \).

Explicit free generators of \( \mathcal{A}_\mu \) for \( \mathfrak{g} = \mathfrak{gl}_N \):

- A. Tarasov, 2000
- Chervov and Talalaev, 2006, 2009

Solution of Vinberg’s problem for any \( \mathfrak{g} \):

- Rybnikov, 2006, regular semi-simple \( \mu \),
- Feigin, Frenkel and Toledano Laredo, 2010, any regular \( \mu \)
Explicit generators

Let $g = \mathfrak{gl}_N$, $E_{ij}$, $i, j \in \{1, \ldots, N\}$ the standard basis of $\mathfrak{gl}_N$. Let $E = (E_{ij})$ and $\mu = (\mu_{ij})$.

Write

$$cdet(-\partial_z + \mu + Ez^{-1}) = \sum_{0 \leq i \leq k \leq N} \phi_k^{(i)}z^{-k+i} \partial_z^{N-k}$$

and

$$tr(-\partial_z + \mu + Ez^{-1})^k = \sum_{0 \leq i \leq k \leq N} \psi_k^{(i)}z^{-k+i} \partial_z^{N-k}$$
Theorem
For any $\mu$ elements $\phi_k^{(i)}$ and $\psi_k^{(i)}$ are two families of generators of a commutative subalgebra $A_\mu$ of $U(\mathfrak{gl}_N)$. If $\mu$ is regular, then the elements of each of these families with $k = 1, ..., N$ and $i = 0, 1, ..., k - 1$ are algebraically independent

Algebraically independent generators of the algebra $A_\mu$ for regular $\mu$: for $\mathfrak{gl}_2$: $\text{tr } E$, $\text{tr } \mu E$, $\text{tr } E^2$
for $\mathfrak{gl}_3$: $\text{tr } E$, $\text{tr } \mu E$, $\text{tr } \mu^2 E$, $\text{tr } E^2$, $\text{tr } \mu E^2$, $\text{tr } E^3$
for $\mathfrak{gl}_4$: $\text{tr } E$, $\text{tr } \mu E$, $\text{tr } \mu^2 E$, $\text{tr } \mu^3 E$, $\text{tr } E^2$, $\text{tr } \mu E^2$,
$2 \, \text{tr } \mu^2 E^2 + \text{tr } (\mu E)^2$, $\text{tr } E^3$, $\text{tr } \mu E^3$, $\text{tr } E^4$